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THREE-DIMENSIONAL INTERACTIONS OF A CIRCULAR CRACK WITH DIPOLES, CENTERS OF DILATATION AND MOMENTS

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Abstract—Exact solutions in elementary functions are derived for the stress intensity factors of a circular crack interacting with various stress sources: dipoles, moments, centers of dilatation and rotation. Such stress sources may model defects like vacancies, foreign particles, dislocations. Locations and orientations of the stress sources with respect to the crack are arbitrary. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Stress intensity factors (SIFs) induced on a circular crack by stress sources like dipoles, moments, centers of dilatation and rotation are derived in elementary functions. Positions and orientations of the stress sources with respect to the crack are arbitrary.

Besides being of interest of their own, these solutions are important for applications, where such stress sources may, under certain conditions, model defects like microcracks, cavities, foreign particles, dislocations, etc.

These solutions are also relevant for the interaction of a crack with an arbitrary system of forces distributed in a small volume. Although such a problem can be handled in a direct way, by simply summing (or integrating) the results for separate stress sources, this would yield exceedingly cumbersome expressions. A substantial simplification is possible if the volume, over which the stress sources are distributed, is small, as compared to its distance from the crack. In this case, the impact of the force system on the crack can be reduced, to within small values of higher order, to impacts of the resultant vector, resultant moment and three mutually orthogonal dipoles.

To avoid making formulas, that are already lengthy, even lengthier, we assume the solid to be isotropic. Generalization to the case of the transversely isotropic solid (provided the crack is parallel to the plane of isotropy) can be obtained, in a straightforward way, using the same method.

Our approach is based on the new method in the potential theory developed by Fabrikant (1989a,b) and applied to the problem of a circular crack.

In the literature, several special cases of a penny-shaped crack interacting with point forces have been considered. In the simplest case when the forces are applied directly at the crack faces, expressions for SIFs follow, in principle, from the results of Galin (1961) on the mathematically similar contact problem and of Uflyand (1965), although the mentioned works do not contain explicit formulas for SIFs. Collins (1962) gave explicit results for SIFs in the axisymmetric case of a point force normal to the crack and applied above the crack center. Kassir and Sih (1975) solved the case of a point force applied above the crack center in the direction tangential to the crack and Bueckner (1987)—the case of a force of arbitrary direction with the point of application in the plane of the crack (outside of it). Finally, Karapetian and Hanson (1994), using the new method of Fabrikant (1989a,b), derived SIFs due to an arbitrary point force in space.

We also mention several related works on somewhat different configurations. Rice (1985a) derived the mode I weight function from which the expression for K_I due to a point force normal to the crack and applied at an arbitrary point in space immediately follows.

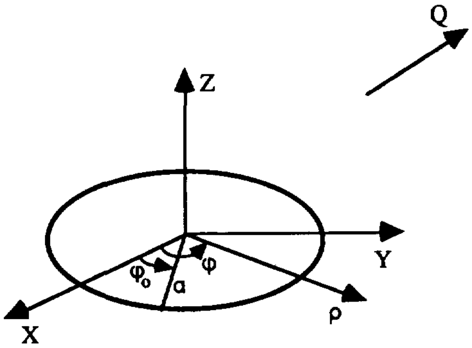

Fig. 1. The configuration of a penny shaped crack and a point force.

(Note that his work also analyzed a more general configuration of a semi-infinite planar crack with a perturbed front line.) Bueckner (1989) gave results for the SIFs in the case of a half-plane crack interacting with a point force of arbitrary direction applied at the point coplanar to the crack. His results were applied by Gao (1989) to the problem of an external circular crack interacting with a point force of arbitrary direction applied at the ligament in the plane of the crack. Rice (1985b) and Hanson (1990, 1992) considered three-dimensional interactions of a crack and a coplanar dislocation loop. Fabrikant *et al.* (1994) solved the problem for the external circular crack in the transversely isotropic material interacting with a symmetric system of two forces normal to the crack and derived K_I due to a point force applied at an arbitrary point.

2. INTERACTION OF AN ARBITRARILY POSITIONED POINT FORCE WITH A CIRCULAR CRACK

We start with the problem of a circular crack interacting with a point force $\mathbf{Q} = (Q_x, Q_y, Q_z)$ of arbitrary direction applied at an arbitrary point (ρ, ϕ, z); the edge of a crack is parametrized by angle ϕ_0 (Fig. 1). Results for SIFs derived recently by Karapetian and Hanson (1994) are transformed here to a simpler form. Namely, they can be reduced, after some algebra, to the following expressions:

$$K_I(\phi_0) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{(a^2 - l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}} \begin{Bmatrix} \operatorname{Re} f_1 \\ \operatorname{Im} f_1 \\ f_2 \end{Bmatrix} \tag{1}$$

$$K_{II}(\phi_0) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{(a^2 - l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}} \operatorname{Re} \begin{Bmatrix} e^{-i\phi_0}(f_3 - f_4) \\ ie^{-i\phi_0}(f_3 + f_4) \\ f_5 \end{Bmatrix} \tag{2}$$

$$K_{III}(\phi_0) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{(a^2 - l_1^2)^{1/2}}{4\pi^2\sqrt{2a}} \operatorname{Im} \begin{Bmatrix} e^{-i\phi_0}(f_3 - f_4) \\ ie^{-i\phi_0}(f_3 + f_4) \\ f_5 \end{Bmatrix} \tag{3}$$

where the elementary functions $f_{1-5}(\rho, \phi, z; \phi_0)$ are introduced (overbar denotes a complex conjugate):

$$f_1 = \frac{1}{\bar{q}}\left[\frac{z}{R^2}\left(2(1-\nu)-\frac{2z^2}{R^2}+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)+\frac{z l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right.$$
$$\left. -(1-2\nu)\frac{a}{\bar{s}(a^2-l_1^2)^{1/2}}\arctan\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)\right] \quad (4)$$

$$f_2 = \frac{1}{R^2}\left[2(1-\nu)+\frac{2z^2}{R^2}-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right] \quad (5)$$

$$f_3 = \frac{1}{2R^2}\left(\frac{2z^2}{R^2}-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-2(2-\nu)\right)+\frac{\nu}{2-\nu}\frac{1}{2s^2}\left[\left(\frac{3a^2}{s^2}-\frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\right)\right.$$
$$\times\left(2\nu-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{z^2 l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\right)-\frac{3z^2 l_2^2 a\rho e^{i(\phi-\phi_0)}}{(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}$$
$$\left.+(1-2\nu)\frac{3a^2(l_2^2-a^2)^{1/2}}{s^3}\arctan\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)\right] \quad (6)$$

$$f_4 = \frac{1}{2\bar{q}^2}\left[\frac{R^2+z^2}{R^2}\left(2\nu-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)+2\nu-\frac{2z^2(R^2-z^2)}{R^4}-\frac{3z^2 l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right.$$
$$+\frac{2}{2-\nu}\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\left(\frac{\nu}{2}+\frac{\bar{s}^2}{l_1^2-a^2+\bar{s}^2}-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)+(1-2\nu)\frac{3(l_2^2-a^2)^{1/2}}{\bar{s}}\arctan\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)$$
$$\left.+\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\left(2\nu+\frac{\nu}{2-\nu}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2 l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\right)\right)\right] \quad (7)$$

$$f_5 = \frac{a}{\bar{s}^2}\left[\frac{z}{R^2}\left(2\nu-\frac{2z^2}{R^2}+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)+\frac{z l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right.$$
$$\left.+(1-2\nu)\frac{a}{\bar{s}(a^2-l_1^2)^{1/2}}\arctan\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)\right]$$
$$+\frac{\nu(1-2\nu)}{2-\nu}\left[\frac{1}{a(a^2-l_1^2)^{1/2}}\left(\arcsin\left(\frac{a}{l_2}\right)-\frac{a^3}{s^3}\arctan\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)\right)\right.$$
$$\left.+\frac{z\rho a^2 e^{i(\phi-\phi_0)}}{s^2(l_2^2-a^2+s^2)(a^2-l_1^2)}\right]-\frac{\nu}{2-\nu}\frac{z\rho e^{i(\phi-\phi_0)}}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\left(1+\frac{2l_2^2}{l_2^2-a^2+s^2}\right). \quad (8)$$

The following notations are used in the formulae above and throughout the text:

$R^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z^2$ (R is the distance between the crack edge and the point of application of the force)

$R_0^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)$ (R_0 is the projection of R onto the crack plane)

$s^2 = a^2-a\rho e^{i(\phi-\phi_0)}, q = \rho e^{i\phi}-ae^{i\phi_0}$ (note that $q\bar{q} = R_0^2$).

In addition, the following two geometric parameters play a key role in the analysis:

$$2l_1 = \sqrt{(a+\rho)^2+z^2}-\sqrt{(a-\rho)^2+z^2}, \quad 2l_2 = \sqrt{(a+\rho)^2+z^2}+\sqrt{(a-\rho)^2+z^2}.$$

Note that l_1 remains bounded as either ρ or z (or both) increase and that the following relation (important for the algebra involved) holds: $a^2z^2 = (a^2-l_1^2)(l_2^2-a^2)$.

We remark that K_{II}, K_{III} due to Q_y can be obtained from those due to Q_x by the replacements $\phi \to \phi+\pi/2$, $\phi_0 \to \phi_0+\pi/2$ and $Q_x \to Q_y$. Conversely, the SIFs due to Q_x are obtained from those due to Q_y by making the same replacements for the angles, although the replacement $Q_y \to -Q_x$ now involves a minus. For K_I, a similar scheme of replacements works in the opposite way, namely, the SIF due to Q_y is obtained from the one due to Q_x by $\phi \to \phi+\pi/2$, $\phi_0 \to \phi_0+\pi/2$ and $Q_x \to -Q_y$ and, conversely, the SIF due to Q_x is obtained from that due to Q_y by $\phi \to \phi+\pi/2$, $\phi_0 \to \phi_0+\pi/2$ and $Q_y \to Q_x$.

It is assumed, throughout this work, that $z \geqslant 0$ (the point of application of the stress source is "above" the crack). Results for $z < 0$ can be obtained from the ones for $z \geqslant 0$ by the following symmetry relations [Kachanov (1993)]: $f_k(-z,\rho,\phi;\phi_0) = f_k(z,\rho,\phi+\pi;\phi_0+\pi)$, with similar relations holding for the functions g_k in the text to follow.

Two previously published results—for the axisymmetrical configuration [Collins (1962), Kassir and Sih (1975)] and the coplanar configuration [Bueckner (1987)]—are recovered as special cases, at $\rho = 0$ and $z = 0$, correspondingly.

An interesting physical observation (that may seem counterintuitive) can be made. If the point of application of Q is in the plane of the crack, then the normal (tangential) component of Q does not generate any mode I (modes II, III) SIFs. This consequence of eqns (1)–(3) can be reconciled with intuition via the reciprocity theorem.

3. INTERACTION OF A CIRCULAR CRACK WITH DIPOLES

A system of two equal and opposite point forces $\mathbf{Q} = (Q_x, Q_y, Q_z)$ and $-\mathbf{Q}$ with points of application lying on the line of $\mathbf{Q}$ and separated by distance h, in the limit of $Q \to \infty$ and $h \to 0$, is called a dipole. The limiting value of the product $\lim_{Q\to\infty, h\to 0} Qh \equiv P$ is called dipole's intensity (assumed to be finite); it is taken to be positive (negative) if the forces are directed away from (towards) each other.

The solution for K_I can be obtained from the solution for the point force by taking the directional derivative of (1) in the direction of $\mathbf{Q}$. Since $Q_k = Q\alpha_k$ ($k = x, y, z$), we have:

$$K_I(\phi_0) = \frac{P}{4\pi^2(1-\nu)\sqrt{2a}} \sum_{m,n=x,y,z} \alpha_n\alpha_m \frac{\partial F_m}{\partial n} \tag{9}$$

where $F_x \equiv (a^2-l_1^2)^{1/2}\,\mathrm{Re} f_1$, $F_y \equiv (a^2-l_1^2)^{1/2}\,\mathrm{Im} f_1$, $F_z \equiv (a^2-l_1^2)^{1/2} f_2$ and α_k are directional cosines between the x, y, z axes and the dipole line (since (9) is quadratic in α_k, it is invariant with respect to the choice of sense along this line).

Calculations yield the following expression for K_I:

$$K_I(\phi_0) = \frac{P(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}}\{\mathrm{Re}\,[\alpha_x^2 g_1+\alpha_y^2 g_2+\alpha_z\alpha_x g_3]$$
$$+\mathrm{Im}\,[\alpha_x\alpha_y g_1-\alpha_x\alpha_y g_2+\alpha_z\alpha_y g_3]+\alpha_z[\alpha_x g_4-\alpha_y g_5+\alpha_z g_6]\} \tag{10}$$

where the elementary functions $g_{1-6}(\rho,\phi,z;\phi_0)$ are given in the Appendix.

Similarly, the solution for K_{II} has the form

$$K_{II}(\phi_0) = \frac{P}{4\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Re}\left(\sum_{m,n=x,y,z} \alpha_n\alpha_m \frac{\partial H_m}{\partial n}\right) \tag{11}$$

where $H_x \equiv (a^2-l_1^2)^{1/2} e^{-i\phi_0}(f_3-f_4)$, $H_y \equiv (a^2-l_1^2)^{1/2} ie^{-i\phi_0}(f_3+f_4)$, $H_z \equiv (a^2-l_1^2)^{1/2} f_5$ and calculations yield

$$K_{II}(\phi_0) = \frac{P(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}} \operatorname{Re}\{e^{-i\phi_0}\alpha_x[\alpha_x(g_7-g_8)-\alpha_y(g_9+g_{10})+\alpha_z(g_{11}-g_{12})]$$

$$+ie^{-i\phi_0}\alpha_y[\alpha_x(g_7+g_8)-\alpha_y(g_9-g_{10})+\alpha_z(g_{11}+g_{12})]+\alpha_z[\alpha_x g_{13}-\alpha_y g_{14}+\alpha_z g_{15}]\} \quad (12)$$

where the elementary functions $g_{7-15}(\rho,\phi,z;\phi_0)$ are given in the Appendix.

The solution for K_{III} is obtained from the one for K_{II} by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im in (12).

Figure 2 illustrates these results.

Coplanar case. If the point of application of the dipole lies in the plane of the crack, the results simplify considerably, although calculations require finding non-trivial limits of the type 0/0.

Two cases should be distinguished: if the point of application of the dipole lies on the crack face, then l_1 should be understood as ρ and l_2 as a; if the dipole is applied outside of the crack, l_1 should be understood as a and l_2 as ρ. For the case when this point is outside of the crack (the case when the mentioned point is on the crack face is analyzed similarly) the results are as follows:

$$K_I(\phi_0)$$

$$= P\frac{(1-2\nu)}{4\pi^2(1-\nu)\sqrt{2a}}\frac{a}{(\rho^2-a^2)^{1/2}}\frac{1}{R_0^2}\Bigg\{\alpha_z^2+(\alpha_x\cos\phi+\alpha_y\sin\phi)^2$$

$$-\alpha_x\alpha_y\left[\frac{\rho^2-a^2}{R_0^2}\left(\sin 2\phi-\frac{a}{\rho}\sin(\phi+\phi_0)\right)-\frac{3}{R_0^2}\operatorname{Im}\left(\frac{q^2(\rho^2-a^2)^{1/2}}{\bar{s}}\right.\right.$$

$$\left.\left.\times\arctan\left(\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right)\right]-\frac{1}{2}(\alpha_x^2-\alpha_y^2)\left[\frac{\rho^2-a^2}{R_0^2}\left(\cos 2\phi-\frac{a}{\rho}\cos(\phi+\phi_0)\right)\right.$$

$$\left.-\frac{3}{R_0^2}\operatorname{Re}\left(\frac{q^2(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\left(\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right)\right]\Bigg\} \quad (13)$$

$$K_{II}(\phi_0)$$

$$= P\frac{(1-2\nu)}{4\pi^2(1-\nu)(2-\nu)\sqrt{2a}}\frac{a}{(\rho^2-a^2)^{1/2}}\frac{1}{R_0^2}\alpha_z\Bigg\{\alpha_x\left[\frac{4\nu-5}{1-2\nu}\cos\phi_0\right.$$

$$-\frac{1}{1-2\nu}\frac{1}{R_0^2}\operatorname{Re}\left(q^2e^{-i\phi_0}\left(3\nu(3-2\nu)-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)+\cos\phi\left(2(1+\nu)\cos(\phi-\phi_0)-\nu\frac{a}{\rho}\right.$$

$$\left.\left.-4\nu\frac{a\rho\sin^2(\phi-\phi_0)}{R_0^2}\right)-\frac{\rho^2-a^2}{R_0^2}\left((1-\nu)\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)-2(1-2\nu)\sin\phi\sin(\phi-\phi_0)\right)\right]$$

$$+\alpha_y\left[\frac{4\nu-5}{1-2\nu}\sin\phi_0-\frac{1}{1-2\nu}\frac{1}{R_0^2}\operatorname{Im}\left(q^2e^{-i\phi_0}\left(3\nu(3-2\nu)-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right.$$

$$+\sin\phi\left(2(1+\nu)\cos(\phi-\phi_0)-\nu\frac{a}{\rho}-4\nu\frac{a\rho\sin^2(\phi-\phi_0)}{R_0^2}\right)$$

$$\left.-\frac{\rho^2-a^2}{R_0^2}\left((1-\nu)\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)+2(1-2\nu)\cos\phi\sin(\phi-\phi_0)\right)\right]\Bigg\} \quad (14)$$

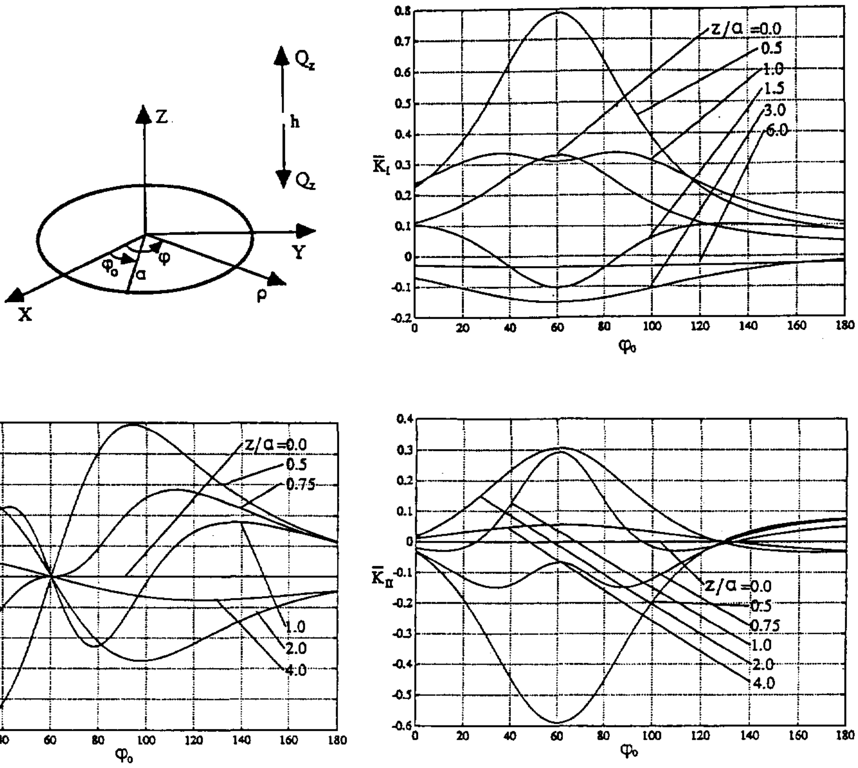

Fig. 2. Dimensionless stress intensity factors $\bar{K}_{I,II,III} = 4\pi^2\sqrt{2}\,a^{5/2}P^{-1}K_{I,II,III}$ around the crack edge due to a dipole oriented in the z-direction and applied at the point $(\rho/a = 2;\, \phi = \pi/3;\, z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

$$
\begin{aligned}
K_{III}(\phi_0) &= P\frac{(1-2\nu)}{4\pi^2(2-\nu)\sqrt{2a}}\,\frac{a}{(\rho^2-a^2)^{1/2}}\,\frac{1}{R_0^2}\,\alpha_z\left\{\alpha_x\left[2\left(1-\frac{\rho^2-a^2}{R_0^2}\right)\cos\phi\,\sin(\phi-\phi_0)\right.\right.\\
&\quad\left.-\frac{4\nu-5}{1-2\nu}\sin\phi_0-\frac{\rho^2-a^2}{R_0^2}\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)-\frac{1}{1-2\nu}\,\frac{1}{R_0^2}\,\mathrm{Im}\left(q^2e^{-i\phi_0}\left(6\nu-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right]\\
&\quad+\alpha_y\left[2\left(1-\frac{\rho^2-a^2}{R_0^2}\right)\sin\phi\,\sin(\phi-\phi_0)+\frac{4\nu-5}{1-2\nu}\cos\phi_0+\frac{\rho^2-a^2}{R_0^2}\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)\right.\\
&\quad\left.\left.+\frac{1}{1-2\nu}\,\frac{1}{R_0^2}\,\mathrm{Re}\left(q^2e^{-i\phi_0}\left(6\nu-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right]\right\}. \qquad (15)
\end{aligned}
$$

In particular, for a dipole in the z-direction, the results are:

$$
K_I(\phi_0) = P\frac{(1-2\nu)}{4\pi^2(1-\nu)\sqrt{2a}}\,\frac{a}{(\rho^2-a^2)^{1/2}}\,\frac{1}{R_0^2},\quad K_{II} = K_{III} = 0. \qquad (16)
$$

Note a similarity between the expression (16) and the one for a circular crack interacting with a coplanar infinitesimal dislocation loop with Burgers' vector **b** in the z-direction [Hanson (1990)], for which

$$K_I = \frac{bE}{4\pi^2(1-v^2)\sqrt{2a}} \frac{a}{(\rho^2-a^2)^{1/2}} \frac{1}{R_0^2}. \tag{17}$$

Comparison of (16) and (17) establishes the equivalence relation between the dipole intensity and the magnitude of Burgers' vector: $P = bE/[(1+v)(1-2v)]$.

4. INTERACTION OF A CIRCULAR CRACK WITH A CENTER OF DILATATION

Three mutually orthogonal dipoles of equal intensity P applied at the same point constitute a center of dilatation (of intensity P). Such a stress source may be used to model a lattice vacancy, an interstitial atom or a foreign particle of the spherical shape.

Note that the stress field induced by such a source is identical to the one generated by a pressurized spherical cavity of radius b in an elastic continuum outside of the cavity (Lamé problem). The equivalence is established by the following relation between the pressure σ on the cavity and the dipole intensity:

$$\sigma = \frac{1-2v}{2\pi(1-v)b^3} P. \tag{18}$$

We consider a center of dilatation of intensity P applied at the point x, y, z and formed by three dipoles in mutually orthogonal directions characterized by directional cosines $(\alpha_x, \alpha_y, \alpha_z)$, $(\beta_x, \beta_y, \beta_z)$ and $(\gamma_x, \gamma_y, \gamma_z)$. The mode I SIF induced on the crack has the form:

$$K_I = \frac{P}{4\pi^2(1-v)\sqrt{2a}} \sum_{m,n=x,y,z} (\alpha_n\alpha_m + \beta_n\beta_m + \gamma_n\gamma_m) \frac{\partial F_m}{\partial n}. \tag{19}$$

Since $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are mutually orthogonal unit vectors, the expression in parentheses constitutes the nm component of the unit tensor, i.e. it equals to Kronecker's delta δ_{mn}. Thus,

$$K_I = \frac{P}{4\pi^2(1-v)\sqrt{2a}} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right). \tag{20}$$

This result is independent of the orientation of the triad $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$; for example, the dipoles can be assumed aligned with the x, y, z directions. It is this invariance that justifies the concept of a center of dilatation.

Equation (20), in principle, solves the problem. However, an alternative, more economical, calculation that yields results in a simpler form can be suggested: we differentiate the point force solution (1) with respect to x, y, z and use the operator $\bar{\Lambda} = \partial/\partial x - i\,\partial/\partial y = e^{-i\phi}(\partial/\partial\rho - (i/\rho)\,\partial/\partial\phi)$ for transition to the cylindrical coordinates. Then

$$K_I = P\frac{1}{4\pi^2(1-v)\sqrt{2a}} \left\{ \frac{\partial}{\partial z}[\sqrt{a^2-l_1^2}f_2] + \operatorname{Re}\bar{\Lambda}[\sqrt{a^2-l_1^2}f_1] \right\}. \tag{21}$$

Similar operations for modes II and III (that involve the operator $\Lambda = \partial/\partial x + i\,\partial/\partial y$) yield all three SIFs:

$$\left.\begin{matrix} K_I(\phi_0) \\ K_{II}(\phi_0) \\ K_{III}(\phi_0) \end{matrix}\right\} = P\frac{(a^2-l_1^2)^{1/2}}{4\pi^2(1-v)\sqrt{2a}} \operatorname{Re}\left\{ \begin{matrix} g_{16} \\ g_{15} + e^{-i\phi_0}(g_{17}-g_{18}) \\ -(1-v)i[g_{15} + e^{-i\phi_0}(g_{17}-g_{18})] \end{matrix} \right\} \tag{22}$$

in terms of elementary functions $g_{15-18}(\rho, \phi, z; \phi_0)$ given in the Appendix.

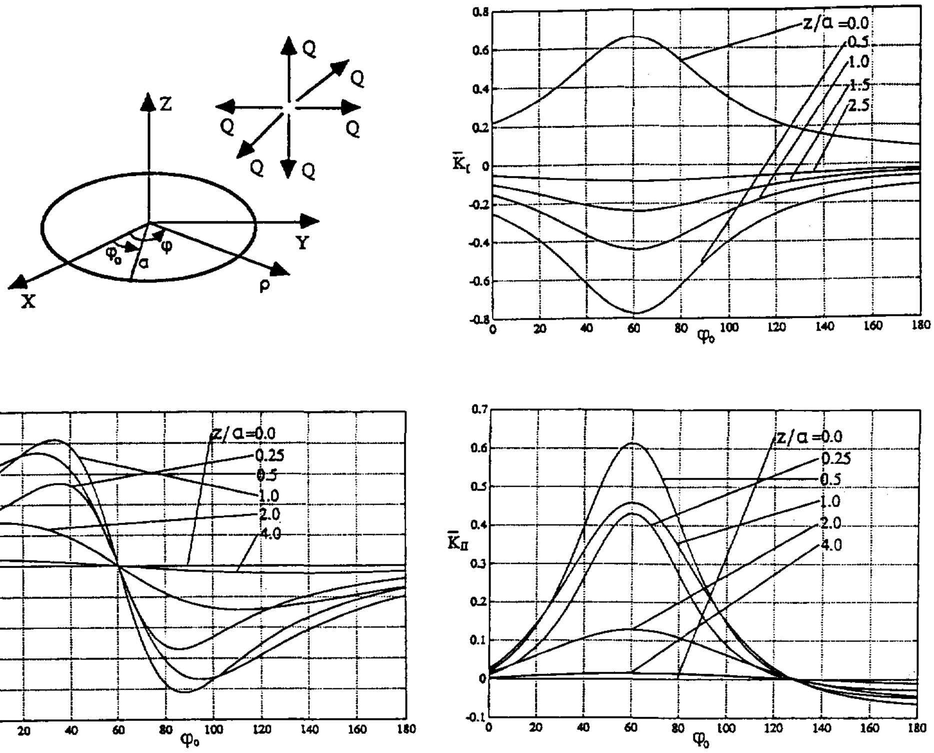

Fig. 3. Dimensionless stress intensity factors $\bar{K}_{I,II,III} = 4\pi^2\sqrt{2}a^{5/2}P^{-1}K_{I,II,III}$ around the crack edge due to a center of dilatation located at the point $(\rho/a = 2; \phi = \pi/3; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

These results are illustrated in Fig. 3.

Coplanar case. In the case when the point of application of the center of dilatation lies in the plane of the crack $z = 0$ (and is outside of the crack),

$$K_I(\phi_0) = \frac{P(1-2\nu)}{2\pi^2(1-\nu)\sqrt{2a}} \frac{a}{(\rho^2-a^2)^{1/2}} \frac{1}{R_0^2}, \quad K_{II} = K_{III} = 0. \tag{23}$$

Note that the result in (23) is twice the value of K_I due to a coplanar dipole in the z direction.

5. INTERACTION OF A CIRCULAR CRACK WITH A MOMENT

A pair of two equal and opposite point forces $\mathbf{Q}$ and $-\mathbf{Q}$, with the lines of action separated by the moment arm h and applied at the points where h intersects these lines, in the limit of $Q \to \infty$ and $h \to 0$, constitutes a concentrated moment. The limiting value of Qh is the intensity of the moment and is denoted by M.

The SIFs due to a moment can be obtained by differentiating the results for the point force in the direction normal to the force, i.e. along the moment arm h. Denoting the directional cosines of h by β_x, β_y, β_z and taking into account that $Q_k = Q\alpha_k$ $(k = x, y, z)$, where α_k are directional cosines of the force direction, we obtain, for the mode I SIF:

$$K_I(\phi_0) = \frac{M}{4\pi^2(1-\nu)\sqrt{2a}} \sum_{m,n=x,y,z} \beta_m \alpha_n \frac{\partial F_n}{\partial m} \tag{24}$$

where functions F_x, F_y, F_z are defined as in (9). Calculations yield the following result in terms of the elementary functions g_{1-6} given in the Appendix:

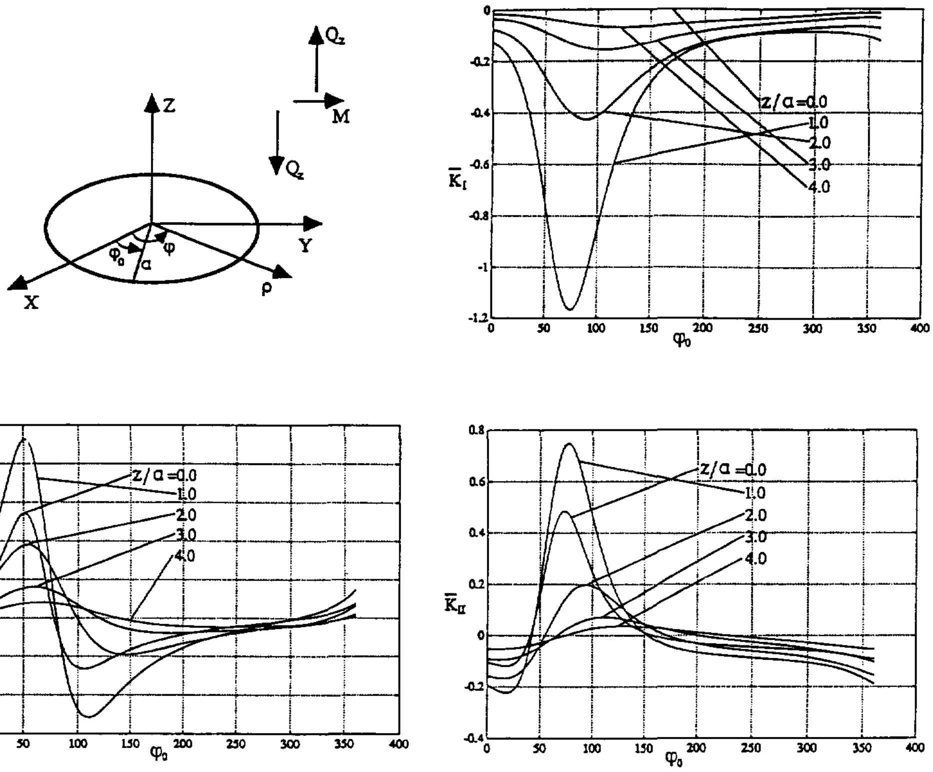

Fig. 4. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}a^{5/2}M^{-1}K_{(I,II,III)}$ around the crack edge due to a moment with the moment axis in the y-direction (produced by a pair of forces in the z-direction) applied at the point ($\rho/a = 2$; $\phi = \pi/3$; z) for several values of z. Poisson's ratio $\nu = 0.3$.

$$K_I(\phi_0) = \frac{M(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}}\{\mathrm{Re}\,[\alpha_x\beta_x g_1+\alpha_y\beta_y g_2+\alpha_x\beta_z g_3]$$

$$+\mathrm{Im}\,[\alpha_y\beta_x g_1-\alpha_x\beta_y g_2+\alpha_y\beta_z g_3]+\alpha_z[\beta_x g_4-\beta_y g_5+\beta_z g_6]\}. \quad (25)$$

For K_{II}, the result can be obtained from (24) by replacing functions F_n by H_n (defined as in (11)) and taking the real part of the sum (in accordance with (2)):

$$K_{II}(\phi_0) = \frac{M(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}}\mathrm{Re}\,\{e^{-i\phi_0}\alpha_x[\beta_x(g_7-g_8)-\beta_y(g_9+g_{10})+\beta_z(g_{11}-g_{12})]$$

$$+ie^{-\phi_0}\alpha_y[\beta_x(g_7+g_8)-\beta_y(g_9-g_{10})+\beta_z(g_{11}+g_{12})]+\alpha_z[\beta_x g_{13}-\beta_y g_{14}+\beta_z g_{15}]\}. \quad (26)$$

The solution for K_{III} is obtained from (26) by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im.

Figure 4 illustrates the results.

Coplanar case. If the moment M is applied in the plane of the crack (outside of the crack), the results are as follows:

$$K_I(\phi_0)$$

$$= \frac{M(1-2\nu)}{4\pi^2(1-\nu)\sqrt{2a}}\frac{a}{(\rho^2-a^2)^{1/2}}\frac{1}{R_0^2}\Bigg\{\alpha_z\beta_z+\alpha_x\beta_x\cos^2\phi+\alpha_y\beta_y\sin^2\phi$$

$$+\frac{1}{2}(\alpha_y\beta_y-\alpha_x\beta_x)\left[\frac{\rho^2-a^2}{R_0^2}\left(\cos 2\phi-\frac{a}{\rho}\cos(\phi+\phi_0)\right)\right.$$

$$-\frac{3}{R_0^2}\mathrm{Re}\left(\frac{q^2(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\Bigg]+\frac{1}{2}(\alpha_y\beta_x+\alpha_x\beta_y)\left[\sin 2\phi-\frac{\rho^2-a^2}{R_0^2}\left(\sin 2\phi\right.\right.$$

$$\left.\left.\left.-\frac{a}{\rho}\sin(\phi+\phi_0)\right)+\frac{3}{R_0^2}\mathrm{Im}\left(\frac{q^2(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right]\right\} \quad (27)$$

$$K_{II}(\phi_0)$$

$$=\frac{M(1-2\nu)}{4\pi^2(1-\nu)(2-\nu)\sqrt{2a}}\frac{a}{(\rho^2-a^2)^{1/2}}\frac{1}{R_0^2}\left\{\frac{4\nu-5}{1-2\nu}\beta_z(\alpha_x\cos\phi_0+\alpha_y\sin\phi_0)\right.$$

$$+\alpha_z(\beta_x\cos\phi+\beta_y\sin\phi)\left[2(1+\nu)\cos(\phi-\phi_0)-\nu\frac{a}{\rho}-4\nu\frac{a\rho\sin^2(\phi-\phi_0)}{R_0^2}\right]$$

$$+\alpha_z\frac{\rho^2-a^2}{R_0^2}\left[2(1-2\nu)(\beta_x\sin\phi-\beta_y\cos\phi)\sin(\phi-\phi_0)-(1-\nu)\beta_x\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)\right.$$

$$\left.-(1-\nu)\beta_y\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)\right]-\beta_z\frac{1}{(1-2\nu)R_0^2}\left[\alpha_x\mathrm{Re}\left(q^2e^{-i\phi_0}\left(3\nu(3-2\nu)-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right.$$

$$\left.+\alpha_y\mathrm{Im}\left(q^2e^{-i\phi_0}\left(3\nu(3-2\nu)-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right]$$

$$+\frac{3(1-\nu)}{R_0^2}\left[(\alpha_z\beta_x-\alpha_x\beta_z)\mathrm{Re}\left(\frac{q^2e^{-i\phi_0}(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right.$$

$$\left.\left.+(\alpha_z\beta_y-\alpha_y\beta_z)\mathrm{Im}\left(\frac{q^2e^{-i\phi_0}(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right]\right\} \quad (28)$$

$$K_{III}(\phi_0)$$

$$=\frac{M(1-2\nu)}{4\pi^2(2-\nu)\sqrt{2a}}\frac{a}{(\rho^2-a^2)^{1/2}}\frac{1}{R_0^2}\left\{\frac{4\nu-5}{1-2\nu}\beta_z(\alpha_y\cos\phi_0-\alpha_x\sin\phi_0)\right.$$

$$+\alpha_z\frac{\rho^2-a^2}{R_0^2}\left[\beta_y\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)-\beta_x\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)\right]$$

$$+2\alpha_z(\beta_x\cos\phi+\beta_y\sin\phi)\left(1-\frac{\rho^2-a^2}{R_0^2}\right)\sin(\phi-\phi_0)$$

$$+\beta_z\frac{1}{(1-2\nu)R_0^2}\left[\alpha_y\mathrm{Re}\left(q^2e^{-i\phi_0}\left(6\nu-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)-\alpha_x\mathrm{Im}\left(q^2e^{-i\phi_0}\left(6\nu-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)\right)\right]$$

$$+\frac{3}{R_0^2}\left[(\alpha_z\beta_x-\alpha_x\beta_z)\mathrm{Im}\left(\frac{q^2e^{-i\phi_0}(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right.$$

$$\left.\left.+(\alpha_y\beta_z-\alpha_z\beta_y)\mathrm{Re}\left(\frac{q^2e^{-i\phi_0}(\rho^2-a^2)^{1/2}}{\bar{s}}\arctan\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)\right]\right\}. \quad (29)$$

6. INTERACTION OF A CIRCULAR CRACK WITH A CENTER OF ROTATION

Center of rotation of intensity M is formed by two mutually orthogonal force pairs applied at the same point and producing moments of the same intensity M in the same direction.

Superimposing the result (24) and the one obtained from (24) by replacements $(\beta_x, \beta_y, \beta_z) \to (\alpha_x, \alpha_y, \alpha_z)$, $(\alpha_x, \alpha_y, \alpha_z) \to (-\beta_x, -\beta_y, -\beta_z)$ yields:

$$K_I(\phi_0) = \frac{M}{4\pi^2(1-\nu)\sqrt{2a}}\left\{(\alpha_y\beta_z-\alpha_z\beta_y)\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right)+(\alpha_z\beta_x-\alpha_x\beta_z)\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\right.$$
$$\left.+(\alpha_x\beta_y-\alpha_y\beta_x)\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right)\right\}. \quad (30)$$

Since the expressions $\alpha_n\beta_m-\alpha_m\beta_n$ in (30) constitute the x, y and z components of the unit vector of the moment direction, we obtain:

$$K_I(\phi_0) = \frac{1}{4\pi^2(1-\nu)\sqrt{2a}}\left\{M_x\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right)+M_y\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)+M_z\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right)\right\}. \quad (31)$$

An important observation is that K_I is expressed solely in terms of vector **M** and does not depend on the exact orientation (in the plane normal to **M**) of the two force pairs that constitute **M**. This invariance justifies the concept of a center of rotation.

The elastic field produced by such a stress source (and its impact on SIFs) is identical to the one (outside of the sphere) in the so-called Robin's problem where a rigid sphere of radius b embedded into an elastic continuum is subjected to a rotation θ. This equivalence is established by the following relation between the rotation θ and the moment M: $\theta = M/8\pi G b^3$, where G is the shear modulus.

K_I can be further expressed in terms of the elementary functions g_{1-5}:

$$K_I(\phi_0) = \frac{(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}}\{M_x(\operatorname{Im} g_3+g_5)+M_y(g_4-\operatorname{Re} g_3)-M_z \operatorname{Im}(g_2+g_1)\}. \quad (32)$$

Similarly, for K_{II} we have:

$$K_{II}(\phi_0) = \frac{(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}}\operatorname{Re}\{M_x[ie^{-i\phi_0}(g_{11}+g_{12})+g_{14}]-M_y[e^{-i\phi_0}(g_{11}-g_{12})-g_{13}]$$
$$-M_z e^{-i\phi_0}[(g_9+g_{10})+i(g_7+g_8)]\}. \quad (33)$$

The solution for K_{III} is obtained from the one for K_{II} by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im in (33).

Figure 5 illustrates the results.

Coplanar case. If the point of application of the center of rotation lies in the plane of the crack $z = 0$ (outside of the crack),

$$K_I = 0 \quad (34)$$

$$K_{II} = \frac{1-2\nu}{4\pi^2(1-\nu)(2-\nu)\sqrt{2a}}\,\frac{a}{(\rho^2-a^2)^{1/2}}\,\frac{1}{R_0^2}\left\{\frac{4\nu-5}{1-2\nu}(M_x\sin\phi_0-M_y\cos\phi_0)\right.$$
$$-(M_x\sin\phi-M_y\cos\phi)\left[2(1+\nu)\cos(\phi-\phi_0)-\nu\frac{a}{\rho}-4\nu\frac{a\rho\sin^2(\phi-\phi_0)}{R_0^2}\right]$$

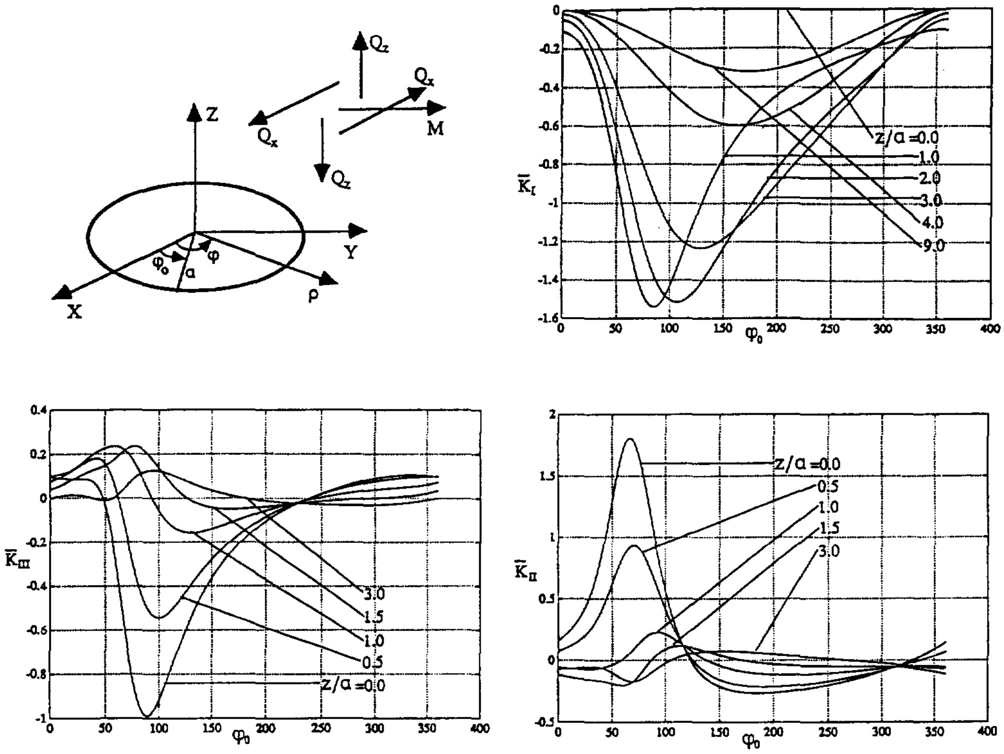

Fig. 5. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}a^{5/2}M^{-1}K_{(I,II,III)}$ around the crack edge due to a center of rotation with the axis of rotation in the y-direction (produced by two pairs of forces, in the z and x-directions) applied at the point $(\rho/a = 2\,;\phi = \pi/3\,;z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

$$+\frac{\rho^2-a^2}{R_0^2}\left[2(1-2\nu)(M_y\sin\phi+M_x\cos\phi)\sin(\phi-\phi_0)-(1-\nu)\left(M_y\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)\right.\right.$$

$$\left.\left.-M_x\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)\right)\right]-\frac{1}{1-2\nu}\,\frac{1}{R_0^2}\,(M_x\,\mathrm{Im}\,h_1-M_y\,\mathrm{Re}\,h_1)\Bigg\} \tag{35}$$

$$K_{III}=\frac{1-2\nu}{4\pi^2(2-\nu)\sqrt{2a}}\;\frac{a}{(\rho^2-a^2)^{1/2}}\;\frac{1}{R_0^2}\left\{\frac{4\nu-5}{1-2\nu}\,(M_x\cos\phi_0+M_y\sin\phi_0)\right.$$

$$-\frac{\rho^2-a^2}{R_0^2}\left[M_x\left(\cos\phi_0-\frac{a}{\rho}\cos\phi\right)+M_y\left(\sin\phi_0-\frac{a}{\rho}\sin\phi\right)\right]$$

$$\left.-2\left(1-\frac{\rho^2-a^2}{R_0^2}\right)(M_x\sin\phi-M_y\cos\phi)\sin(\phi-\phi_0)+\frac{1}{1-2\nu}\;\frac{1}{R_0^2}\,(M_x\,\mathrm{Re}\,h_2+M_y\,\mathrm{Im}\,h_2)\right\} \tag{36}$$

where $h_{1,2}(\rho,\phi\,;\phi_0)$ are introduced:

$$h_1=q^2\mathrm{e}^{-i\phi_0}\left[3\nu(3-2\nu)+6(1-\nu)(1-2\nu)\,\frac{(\rho^2-a^2)^{1/2}}{\bar{s}}\,\arctan\left(\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)-\frac{a\mathrm{e}^{i(\phi-\phi_0)}}{\rho}\right]$$

$$h_2=q^2\mathrm{e}^{-i\phi_0}\left[6\nu+6(1-2\nu)\,\frac{(\rho^2-a^2)^{1/2}}{\bar{s}}\,\arctan\left(\frac{\bar{s}}{(\rho^2-a^2)^{1/2}}\right)-\frac{a\mathrm{e}^{i(\phi-\phi_0)}}{\rho}\right].$$

7. INTERACTION OF A CRACK WITH A SYSTEM OF FORCES DISTRIBUTED IN A SMALL VOLUME

Interaction of a crack with several stress sources (or continuous distribution of them) can, in principle, be analyzed by direct summation (or integration) of the results for SIFs over all the stress sources. However, this may lead to very lengthy expressions or to integrals that cannot be expressed in any standard functions.

We consider a special case when the system of forces is distributed over a volume V which is *small* as compared to its distance from the crack. Then, to within small values of higher order, the results can be obtained in a much simpler way, by reducing the impact of the force system to the ones of the resultant vector, resultant moment and three mutually orthogonal dipoles.

Thus, we consider a crack interacting with a system of point forces $\mathbf{Q}^1, \ldots \mathbf{Q}^N$ applied at the points $O_1, \ldots O_N$ located in a neighbourhood of the point O. The size of this neighbourhood is small as compared to the distance between the crack and the point O (continuous distribution of forces can be considered by changing summations to integrations in the formulas to follow.)

In an infinite linear elastic solid such a system of forces can be represented by a superposition of the following "elementary" sources [see, for example, Lur'e (1964)]:

1. Point force—the resultant vector of $\mathbf{Q}^1, \ldots \mathbf{Q}^N$:

$$\mathbf{Q} = \Sigma \mathbf{Q}^{(i)}. \tag{37}$$

2. The resultant moment about O:

$$\mathbf{M}_o = \Sigma \mathbf{r}^{(i)} \times \mathbf{Q}^{(i)} \tag{38}$$

where $\mathbf{r}^{(i)}$ is the position vector of O_i with respect to O.

3. Three mutually orthogonal dipoles of the intensities P_1, P_2, P_3 and directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ that are the eigenvalues and eigenvectors of the symmetric second rank tensor ("force system tensor")

$$\mathbf{P} = \tfrac{1}{2}\Sigma(\mathbf{Q}^{(i)}\mathbf{r}^{(i)} + \mathbf{r}^{(i)}\mathbf{Q}^{(i)}) \tag{39}$$

where $\mathbf{Qr}$, $\mathbf{rQ}$ are dyadic products of vectors. (In the principal representation $\mathbf{P} = P_1\mathbf{e}_1\mathbf{e}_1 + P_2\mathbf{e}_2\mathbf{e}_2 + P_3\mathbf{e}_3\mathbf{e}_3$, each term corresponding to a dipole.) This system of three dipoles can be further decomposed into a sum of the "hydrostatic" and deviatoric parts:

$$\mathbf{P} = \tfrac{1}{3}J_1\mathbf{I} + (\mathbf{P} - \tfrac{1}{3}J_1\mathbf{I}) \tag{40}$$

where $J_1(\mathbf{P}) = \Sigma\mathbf{Q}^{(i)}\cdot\mathbf{r}^{(i)}$ is the linear invariant of $\mathbf{P}$. The first term in (40) corresponds to a center of expansion, whereas the second term reduces to three dipoles of intensities $P_1 - \langle P \rangle$, $P_2 - \langle P \rangle$, $P_3 - \langle P \rangle$ where $\langle P \rangle = (P_1 + P_2 + P_3)/3$.

These representations are valid to within terms of a higher order with respect to the small parameter (*size of* V)/(*distance from* V *to the point of observation*). In presence of a crack, the same representations hold for the impact of the force system on SIFs, provided *size of* $V \ll$ *distance from* V *to the crack*. Thus, the results derived in the preceding sections constitute a full system of solutions: SIFs due to an arbitrary system of forces distributed in a small volume can be expressed in terms of these solutions.

Note that SIFs due to the resultant vector (37) decrease as (*distance from* V *to the crack*)$^{-2}$ whereas SIFs due to (38, 39) decrease as (*distance from* V *to the crack*)$^{-3}$.

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APPENDIX

Functions $g_k = g_k(\rho, \phi, z ; \phi_0)$ are given here. Functions g_2, g_5, g_9, g_{10} and g_{14} are obtained, respectively, from the functions g_1, g_4, g_7, g_8 and g_{13} given below by the replacements $(\phi, \phi_0) \to (\phi+\pi/2, \phi_0+\pi/2)$ everywhere, including q.

$$g_1 = \frac{1}{(a^2-l_1^2)^{1/2}}\left[\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)(a^2-l_1^2)^{1/2}f_1\right]$$

$$= \frac{z}{\bar{q}R^2}\left[\frac{4z^2(\rho\cos\phi - a\cos\phi_0)}{R^4} + \frac{\rho\cos\phi}{l_2^2-l_1^2}\frac{2z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2} - \left(\frac{2(\rho\cos\phi - a\cos\phi_0)}{R^2} + \frac{\rho\cos\phi}{l_2^2-l_1^2}\right.\right.$$

$$\left.\left. + \frac{1}{\bar{q}}\right)\left(2(1-\nu) - \frac{2z^2}{R^2} + \frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)\right] - \frac{zl_2^2}{\bar{q}(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\left[\frac{1}{\bar{q}} - \frac{ae^{i\phi_0}}{l_2^2-a^2+\bar{s}^2}\right.$$

$$\left. - \frac{\rho\cos\phi}{l_2^2-l_1^2}\left(1 - \frac{2a^2}{l_2^2} - \frac{4(l_2^2-a^2)}{l_2^2-l_1^2} + (1-2\nu)\frac{l_2^2-l_1^2}{l_2^2-\rho^2} + \frac{2\bar{s}^2}{l_2^2-a^2+\bar{s}^2}\right)\right]$$

$$+ \frac{1-2\nu}{2\bar{q}^2 z}\left[\frac{3(l_2^2-a^2)^{1/2}}{\bar{s}}\arctan\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right) - \frac{l_2^2-a^2}{l_2^2-a^2+\bar{s}^2}\right]$$

$$g_3 = \frac{1}{(a^2-l_1^2)^{1/2}}\frac{\partial}{\partial z}[(a^2-l_1^2)^{1/2}f_1]$$

$$= \frac{1}{R^2\bar{q}}\left[\left(1 + \frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{2z^2}{R^2}\right)\left(2(1-\nu) + \frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{2z^2}{R^2}\right) - \frac{4z^2}{R^2}\left(1 - \frac{z^2}{R^2}\right)\right.$$

$$\left. + \frac{2z^2\rho^2(a^2-\rho^2-z^2)}{(l_2^2-l_1^2)^3}\right] + \frac{l_2^2}{\bar{q}(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\left(2(1-\nu) + \frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{4z^2 l_1^2}{(l_2^2-l_1^2)^2}\right.$$

$$\left. - \frac{2z^2 l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right)$$

$$g_4 = \frac{1}{(a^2-l_1^2)^{1/2}}\left[\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)(a^2-l_1^2)^{1/2}f_2\right]$$

$$= \frac{1}{R^2}\Bigg[\frac{2\rho\cos\phi}{l_2^2-l_1^2}\left(\frac{z^2}{R^2} - \frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\right)$$

$$- \left(\frac{\rho\cos\phi}{l_2^2-l_1^2} + \frac{2(\rho\cos\phi - a\cos\phi_0)}{R^2}\right)\left(2(1-\nu) + \frac{4z^2}{R^2} - \frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)\Bigg]$$

$$g_6 = \frac{1}{(a^2-l_1^2)^{1/2}}\frac{\partial}{\partial z}[(a^2-l_1^2)^{1/2}f_2]$$

$$= \frac{1}{R^2 z}\left[\left(2\nu - \frac{4z^2}{R^2}\right)\left(\frac{2z^2}{R^2} - \frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right) + \frac{\rho^2-l_1^2}{l_2^2-l_1^2} + \frac{z^2\rho^2(3l_2^2+l_1^2-4a^2)}{(l_2^2-l_1^2)^3}\right]$$

$$g_7 = \frac{1}{(a^2-l_1^2)^{1/2}}\left[\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)(a^2-l_1^2)^{1/2}f_3\right]$$

$$= \frac{1}{R^2}\Bigg[\frac{2(\rho\cos\phi - a\cos\phi_0)}{R^2}\left(2-\nu-\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)} - \frac{2z^2}{R^2}\right) + \frac{\rho\cos\phi}{l_2^2-l_1^2}\Bigg(2-\nu+\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}$$

$$- \frac{z^2}{R^2} - \frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\Bigg)\Bigg] - \frac{\nu}{2-\nu}\,\frac{1}{s^2}\Bigg\{\frac{5}{2q}\Bigg[(1-2\nu)\frac{3a^2(l_2^2-a^2)^{1/2}}{2s^3}\arctan\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)$$

$$- \frac{z^2 l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\left(\frac{3a^2}{2s^2} + \frac{a^2}{l_2^2-a^2+s^2} + \frac{4s^2 a\rho e^{i(\phi-\phi_0)}}{5(l_2^2-a^2+s^2)^2}\right) + \left(\nu - \frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right)\Bigg(\frac{3a^2}{s^2}$$

$$- \frac{a^2}{l_2^2-a^2+s^2} - \frac{2s^2 a\rho e^{i(\phi-\phi_0)}}{5(l_2^2-a^2+s^2)^2}\Bigg)\Bigg] + \frac{\rho\cos\phi}{l_2^2-l_1^2}\Bigg[\left(\frac{3a^2}{s^2} - \frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\right)\left(\frac{2z^2 l_2^2}{(l_2^2-l_1^2)^2} - \frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right)$$

$$+ \nu\frac{s^2}{l_2^2-a^2+s^2}\left(1 + \frac{2l_2^2}{l_2^2-a^2+s^2}\right) + \frac{z^2 l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\Bigg(\frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\Bigg(\frac{s^2}{l_2^2}$$

$$+ \frac{4s^2}{l_2^2-a^2+s^2}\Bigg) + \left(\frac{3a^2}{2s^2} + \frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\right)\left(\frac{\rho^2-l_1^2}{l_2^2-\rho^2} - \frac{4(l_2^2-a^2)}{l_2^2-l_1^2}\right)\Bigg)\Bigg]\Bigg\}$$

$$g_8 = \frac{1}{(a^2-l_1^2)^{1/2}}\left[\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)(a^2-l_1^2)^{1/2}f_4\right]$$

$$= \frac{1}{\bar{q}^2}\Bigg\{\frac{2z^2(\rho\cos\phi - a\cos\phi_0)}{R^4}\left(1-\nu-\frac{2z^2}{R^2} + \frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right) - \nu\left(\frac{\rho e^{i\phi}}{l_2^2-a^2+s^2} - \frac{l_2^2\bar{q}}{(l_2^2-a^2+s^2)^2}\right)$$

$$- \frac{2}{2-\nu}\,\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\Bigg[\frac{e^{i\phi}}{2\rho}\left(\frac{2a^2}{\bar{s}^2} - \frac{ae^{i(\phi-\phi_0)}}{\rho} - \frac{2a\rho e^{i(\phi-\phi_0)}}{l_1^2-a^2+\bar{s}^2} - \frac{(a^2-l_1^2)a\rho e^{i(\phi-\phi_0)}}{(l_1^2-a^2+\bar{s}^2)^2}\right)$$

$$+ \frac{\rho\cos\phi}{l_2^2-l_1^2}\left(\left(\frac{1}{2} - \frac{(a^2-l_1^2)(l_2^2+l_1^2)}{(l_2^2-l_1^2)l_1^2}\right)\left(\frac{\bar{s}^2}{l_1^2-a^2+\bar{s}^2} - \frac{ae^{i(\phi-\phi_0)}}{\rho}\right) + \frac{(a^2-l_1^2)\bar{s}^2}{(l_1^2-a^2+\bar{s}^2)^2}\right)\Bigg]$$

$$+ \frac{\nu}{2-\nu}\,\frac{z^2 l_2^2\bar{q}\rho e^{i\phi}}{(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}\Bigg[\frac{2}{\bar{q}} - \frac{2ae^{i\phi_0}}{l_2^2-a^2+s^2} + \frac{ae^{i\phi_0}}{\bar{q}\rho e^{i\phi}}\left(1-\frac{\rho^2}{2l_2^2}\right) + \frac{\rho\cos\phi}{l_2^2-l_1^2}\Bigg(\frac{4(l_2^2-a^2)}{l_2^2-a^2+s^2}$$

$$+ \frac{4(l_2^2-a^2)}{l_2^2-l_1^2} - \frac{2(l_2^2-a^2)}{l_2^2} - (5-2\nu)\frac{l_2^2-l_1^2}{l_2^2-\rho^2}\Bigg)\Bigg] + \left(\frac{2}{\bar{q}} + \frac{\rho\cos\phi}{l_2^2-l_1^2}\right)\Bigg[\frac{z^2}{R^2}\left(1-\nu-\frac{z^2}{R^2}+\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right)$$

$$- 2\nu + \frac{\rho^2-l_1^2}{l_2^2-l_1^2}\left(\frac{1-\nu}{2-\nu} - \frac{\nu}{2(2-\nu)}\,\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\right)\Bigg] - \frac{\rho\cos\phi}{l_2^2-l_1^2}\Bigg[\frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\Bigg(\frac{2(1-\nu)}{2-\nu} + \frac{z^2}{R^2}$$

$$- \frac{\nu}{2-\nu}\,\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\Bigg) + \nu\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2} + \frac{\nu}{2-\nu}\left(\frac{2z^2 l_2^2}{(l_2^2-a^2+s^2)^2} - \frac{\rho^2-l_1^2}{l_2^2-a^2+s^2}\right)\Bigg]$$

$$+ \frac{\rho\cos\phi}{l_2^2-l_1^2}\,\frac{3z^2 l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\left(\frac{3}{2} + \frac{a^2}{l_2^2} + \frac{2(l_2^2-a^2)}{l_2^2-l_1^2}\right) + \frac{3(l_2^2-a^2)}{l_2^2-a^2+\bar{s}^2}\left(\frac{1}{2\bar{q}} - \frac{\rho\cos\phi}{l_2^2-l_1^2}\right)\Bigg[\frac{1-2\nu}{2}$$

$$+\frac{l_2^2-\rho^2}{l_2^2-l_1^2}\biggl(2+\frac{\bar{s}^2}{l_2^2-a^2+\bar{s}^2}\biggr)\biggr]-(1-2\nu)\frac{15(l_2^2-a^2)^{1/2}}{4\bar{q}\bar{s}}\arctan\biggl(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\biggr)\biggr\}$$

$$g_{11}=\frac{1}{(a^2-l_1^2)^{1/2}}\frac{\partial}{\partial z}[(a^2-l_1^2)^{1/2}f_3]$$

$$=\frac{1}{R^2z}\biggl[\biggl(\frac{2z^2}{R^2}-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggr)\biggl(3-\nu-\frac{2z^2}{R^2}\biggr)+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggl(\frac{l_2^2-a^2+z^2}{2(l_2^2-l_1^2)}+\frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\biggr)\biggr]$$

$$+\frac{\nu}{2-\nu}\frac{1}{s^2z}\biggl[\biggl(\frac{3a^2}{s^2}-\frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\biggr)\biggl(\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggl(\nu-\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\biggr)-\frac{z^2\rho^2(a^2-\rho^2-z^2)}{(l_2^2-l_1^2)^3}\biggr)$$

$$-\frac{3z^2l_2^2a^2}{2s^2(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\biggl(3-2\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{4z^2l_1^2}{(l_2^2-l_1^2)^2}-\frac{2z^2l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\biggr)$$

$$-\frac{2z^2l_2^2a\rho e^{i(\phi-\phi_0)}}{(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}\biggl(1-\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2l_1^2}{(l_2^2-l_1^2)^2}-\frac{2z^2l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\biggr)$$

$$+(1-2\nu)\frac{3a^2(l_2^2-a^2)^{1/2}}{2s^3}\arctan\biggl(\frac{s}{(l_2^2-a^2)^{1/2}}\biggr)\biggr]$$

$$g_{12}=\frac{1}{(a^2-l_1^2)^{1/2}}\frac{\partial}{\partial z}[(a^2-l_1^2)^{1/2}f_4]$$

$$=\frac{1}{\bar{q}^2z}\biggl\{\biggl(1-\frac{z^2}{R^2}\biggr)\biggl[\biggl(\frac{2z^2}{R^2}-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggr)\biggl(\nu+\frac{2z^2}{R^2}\biggr)-\frac{2z^2}{R^2}\biggr]+\nu\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggl(3+\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\biggr)$$

$$-\frac{z^2}{R^2}\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggl(\frac{1}{2}+\frac{l_2^2-\rho^2}{2(l_2^2-l_1^2)}-\frac{2z^2l_2^2}{(l_2^2-l_1^2)^2}\biggr)+\frac{2}{2-\nu}\frac{\bar{s}^2}{(l_1^2-a^2+\bar{s}^2)^2}\frac{z^2l_1^2(\rho^2-l_1^2)}{(l_2^2-l_1^2)^2}$$

$$-\frac{3l_2^2z^2}{2(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\biggl(3-2\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{4z^2l_1^2}{(l_2^2-l_1^2)^2}-\frac{2z^2l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\biggr)$$

$$-\frac{\nu}{2-\nu}\frac{2z^2l_2^2\bar{q}\rho e^{i\phi}}{(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}\biggl(3-\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2l_1^2}{(l_2^2-l_1^2)^2}-\frac{2z^2l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\biggr)$$

$$+\biggl[\frac{2}{2-\nu}\biggl(\frac{\bar{s}^2}{l_1^2-a^2+\bar{s}^2}-\frac{ae^{i(\phi-\phi_0)}}{\rho}\biggr)+\frac{\nu}{2-\nu}\biggl(1+\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\biggr)-1\biggr]\biggl[\frac{(\rho^2-l_1^2)^2}{2(l_2^2-l_1^2)^2}$$

$$+\frac{z^2\rho^2(a^2-\rho^2-z^2)}{(l_2^2-l_1^2)^3}\biggr]+(1-2\nu)\frac{3(l_2^2-a^2)^{1/2}}{2\bar{s}}\arctan\biggl(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\biggr)\biggr\}$$

$$g_{13}=\frac{1}{(a^2-l_1^2)^{1/2}}\biggl[\biggl(\cos\phi\frac{\partial}{\partial\rho}-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\biggr)(a^2-l_1^2)^{1/2}f_5\biggr]$$

$$=\frac{a}{\bar{s}^2z}\biggl\{\frac{1-2\nu}{2\bar{q}}\biggl[\frac{l_2^2-a^2}{l_2^2-a^2+\bar{s}^2}-\frac{3(l_2^2-a^2)^{1/2}}{\bar{s}}\arctan\biggl(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\biggr)\biggr]+\frac{z^2}{R^2}\biggl[\frac{\rho\cos\phi}{l_2^2-l_1^2}\frac{2z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}$$

$$+\frac{4z^2(\rho\cos\phi-a\cos\phi_0)}{R^4}-\biggl(\frac{2(\rho\cos\phi-a\cos\phi_0)}{R^2}+\frac{\rho\cos\phi}{l_2^2-l_1^2}+\frac{1}{\bar{q}}\biggr)\biggl(2\nu-\frac{2z^2}{R^2}+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\biggr)\biggr]$$

$$+\frac{z^2l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\biggl[\frac{\rho\cos\phi}{l_2^2-l_1^2}\biggl(1-\frac{2a^2}{l_2^2}-(1-2\nu)\frac{l_2^2-l_1^2}{l_2^2-\rho^2}-\frac{4(l_2^2-a^2)}{l_2^2-l_1^2}+\frac{2\bar{s}^2}{l_2^2-a^2+\bar{s}^2}\biggr)$$

$$-\frac{1}{\bar{q}}\biggl(1+\frac{\bar{s}^2}{l_2^2-a^2+\bar{s}^2}\biggr)\biggr]\biggr\}-\frac{\nu(1-2\nu)}{2-\nu}\frac{3a}{2qs^2z}\biggl[\frac{l_2^2-a^2}{l_2^2-a^2+s^2}-\frac{(l_2^2-a^2)^{1/2}}{s}\arctan\biggl(\frac{s}{(l_2^2-a^2)^{1/2}}\biggr)\biggr]$$

$$-\frac{\nu}{2-\nu}\frac{zl_2^2\rho e^{i(\phi-\phi_0)}}{(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}\biggl\{\frac{1-2\nu}{q}\frac{l_2^2-l_1^2}{l_2^2-\rho^2}+\frac{3}{\rho e^{i\phi}}+\frac{4ae^{-i\phi_0}}{l_2^2-a^2+s^2}$$

$$-\frac{\rho\cos\phi}{l_2^2-l_1^2}\biggl[\frac{4a\rho e^{i(\phi-\phi_0)}}{l_2^2-a^2+s^2}\frac{\rho^2-l_1^2}{\rho^2}-\biggl(3-\frac{a\rho e^{i(\phi-\phi_0)}}{l_2^2}\biggr)\biggl(1-(1-2\nu)\frac{l_2^2-l_1^2}{l_2^2-\rho^2}$$

$$-\frac{4(l_2^2-a^2)}{l_2^2-l_1^2}-\frac{2(l_2^2-a^2)}{l_2^2-a^2+s^2}\Bigg)\Bigg]\Bigg\}$$

$$g_{15}=\frac{1}{(a^2-l_1^2)^{1/2}}\frac{\partial}{\partial z}[(a^2-l_1^2)^{1/2}f_5]$$

$$=\frac{a}{R^2\bar{s}^2}\left[\left(1+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2}{R^2}\right)\left(2\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2}{R^2}\right)-\frac{4z^2}{R^2}\left(1-\frac{z^2}{R^2}\right)+\frac{2z^2\rho^2(a^2-\rho^2-z^2)}{(l_2^2-l_1^2)^3}\right]$$

$$+\frac{l_2^2a}{\bar{s}^2(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\left(2\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{4z^2l_1^2}{(l_2^2-l_1^2)^2}-\frac{2z^2l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right)$$

$$-\frac{\nu}{2-\nu}\frac{\rho e^{i(\phi-\phi_0)}}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\Bigg[\left(2\nu+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)\left(1+\frac{2l_2^2}{l_2^2-a^2+s^2}\right)$$

$$-\frac{2z^2}{l_2^2-l_1^2}\left(\frac{l_2^2+l_1^2}{l_2^2-l_1^2}+\frac{l_2^2(l_2^2+3l_1^2)}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}+\frac{4l_2^4}{(l_2^2-a^2+s^2)^2}\right)\Bigg]$$

$$g_{16}=\frac{1}{R^2z}\left[(1-2\nu)\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{4z^2}{R^2}\right)+\frac{2z^2}{R^2}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2}+\frac{z^2\rho e^{-i\phi}}{\bar{q}(l_2^2-l_1^2)}\right)+\frac{z^2\rho^2(3l_2^2+l_1^2-4a^2)}{(l_2^2-l_1^2)^3}\right]$$

$$+\frac{\rho e^{-i\phi}}{\bar{q}z(l_2^2-l_1^2)}\Bigg[\left(\frac{l_2^2-a^2}{l_2^2-a^2+\bar{s}^2}-\frac{z^2}{R^2}\right)\left(2(1-\nu)+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\right)$$

$$-\frac{2(l_2^2-a^2)}{l_2^2-a^2+\bar{s}^2}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2}+\frac{z^2l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right)\Bigg]$$

$$g_{17}=\frac{q}{R^4}\left(2(2-\nu)-\frac{4z^2}{R^2}+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)+\frac{\rho e^{i\phi}}{R^2(l_2^2-l_1^2)}\left[2-\nu-\frac{z^2}{R^2}+\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}-\frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}\right]$$

$$-\frac{\nu}{2-\nu}\frac{\rho e^{i\phi}}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\Bigg[3\nu+\frac{a\rho e^{i(\phi-\phi_0)}}{s^2}\left(2\nu+\frac{z^2}{l_2^2-l_1^2}\right)-\left(1+\frac{2a^2}{s^2}\right)\Bigg(\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}$$

$$-\frac{2z^2l_2^2}{(l_2^2-l_1^2)^2}\Bigg)+\frac{(l_2^2-a^2)a\rho e^{i(\phi-\phi_0)}}{s^2(l_2^2-a^2+s^2)}\Bigg(1-2\nu+\frac{3(a^2-l_1^2)}{l_2^2-l_1^2}-\frac{2z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}$$

$$-\frac{4z^2l_2^2}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\Bigg)\Bigg]$$

$$g_{18}=\frac{z^2}{\bar{q}R^4}\left(2(1-\nu)-\frac{4z^2}{R^2}+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\right)+\frac{\nu}{2-\nu}\frac{2z^2l_2^2}{\bar{q}(l_2^2-a^2+s^2)^2(l_2^2-l_1^2)}\Bigg(1+(2-\nu)\frac{\rho^2}{\rho^2-l_1^2}$$

$$+\frac{(\rho^2-l_1^2)(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}-\frac{2l_2^2(l_2^2-\rho^2)}{(l_2^2-a^2+s^2)(l_2^2-l_1^2)}\Bigg)-\frac{e^{-i\phi}}{\bar{q}^2}\frac{\rho}{l_2^2-l_1^2}\Bigg\{\frac{2}{2-\nu}\frac{z^2l_1^2\bar{s}^2}{(l_1^2-a^2+\bar{s}^2)^2(l_2^2-l_1^2)}$$

$$+\nu\frac{3\bar{s}^2}{l_2^2-a^2+\bar{s}^2}+\nu\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}-\left(1-\frac{z^2}{R^2}\right)\left(\nu+\frac{z^2}{R^2}\right)-\Bigg[\frac{2}{2-\nu}\left(\frac{\bar{s}^2}{l_1^2-a^2+\bar{s}^2}-\frac{ae^{i(\phi-\phi_0)}}{\rho}\right)$$

$$+\frac{\nu}{2-\nu}\left(1+\frac{\bar{q}\rho e^{i\phi}}{l_2^2-a^2+s^2}\right)-\left(1+\frac{z^2}{R^2}\right)\Bigg]\left(\frac{z^2(l_2^2+l_1^2)}{(l_2^2-l_1^2)^2}-\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right)$$

$$-\frac{3z^2l_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\left[\frac{\rho^2+z^2-a^2}{l_2^2-l_1^2}-\frac{\rho^2-l_1^2}{2(l_2^2-\rho^2)}+\frac{\rho^2-l_1^2}{l_2^2-a^2+\bar{s}^2}\frac{ae^{-i(\phi-\phi_0)}}{\rho}\right]\Bigg\}.$$